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OPTIMAL DISCOUNTED LINEAR CONTROL OF THE WIENER PROCESS.(U)
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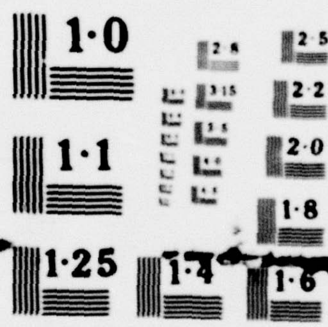

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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subject to $dx_t = u_t(x)dt + dw_t$, $x_0 = x$; $|u_t| \leq 1$, (w_t) a

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20. Abstract continued.

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u_t^*(x) &= -1, & x_t &> b \\
&= 0, & |x_t| &\le b \\
&= 1, & x_t &< -b
\end{aligned}
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for some switching point $b > 0$, characterized in terms of the function $\phi(\cdot)$ through a transcendental equation.

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OPTIMAL DISCOUNTED LINEAR CONTROL OF THE WIENER PROCESS⁺

by

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September 1979

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OPTIMAL DISCOUNTED LINEAR CONTROL OF THE WIENER PROCESS[+]

Ioannis Karatzas

ABSTRACT

The following stochastic control problem is considered: to minimize the discounted expected total cost

$$J(x;u) = E\int_0^\infty e^{-\alpha t}[\phi(x_t) + |u_t(x)|]dt$$

subject to $dx_t = u_t(x)dt + dw_t$, $x_0 = x$; $|u_t| \leq 1$, (w_t) a Wiener process, $\alpha > 0$. All bounded by unity, measurable and nonanticipative functionals $u_t(x)$ of the state process (x_t) are admissible as controls. It is proved that the optimal law is of the form

$$\begin{aligned} u_t^*(x) &= -1, & x_t &> b \\ &= 0, & |x_t| &\leq b \\ &= 1, & x_t &< -b \end{aligned}$$

for some switching point $b > 0$, characterized in terms of the function $\phi(\cdot)$ through a transcendental equation.

1. INTRODUCTION AND SUMMARY

We consider the problem of discounted optimal control of the Wiener process ~~$\{w_t;\ t \geq 0\}$~~ transformed by the action of a non-anticipative control into the "state process" ~~$\{x_t;\ t \geq 0\}$.~~ The latter satisfies the "state equation" ~~dx_t~~ $= u_t(x)dt + dw_t$, $t \geq 0$; x_0 ~~$= x$~~ on an appropriate probability space.

There is a cost ~~$\phi(x_t)$~~ per unit time for being in the wrong state ~~x_t~~ which ~~where~~ ~~$\phi(\cdot)$~~ is an even, uniformly convex function on the reals whose second derivative is decreasing with distance from the origin. There is also a cost ~~$|u_t|$~~ per unit time for using the control, ~~u_t~~ Both costs are discounted in time by a ~~the~~ factor $e^{-\alpha t}$, α ~~> 0~~ and the control action is limited; ~~$|u_t| \leq 1$, a.s.~~ The controller has to choose a law ~~$u_t(x)$~~ as a nonanticipative, measurable functional of the state process with values ~~in~~ $[-1,1]$, so as to minimize the expected discounted total cost. → next page

The "physically obvious" law is to push with full force (to the right direction) if x_t is outside a certain neighbourhood of the origin, while to exert no control at all if x_t is in this neighbourhood:

$$
\begin{aligned}
u_t^*(x) &= -1, & x_t &> b \\
&= 0, & |x_t| &\leq b \qquad (1.1)\\
&= 1, & x_t &< -b
\end{aligned}
$$

Optimality of this law is proved and the cutoff point b separating the active region from the dead zone is characterized in terms of the function $\phi(\cdot)$ through the transcendental equation (3.11). Existence and uniqueness of a solution to the above equation is proved by making use of the aforementioned properties of $\phi(\cdot)$. It is an interesting problem to relax these assumptions in order to allow cost functions with general polynomial or even exponential growth.

General existence results for the problem of discounted stochastic control were given by Kushner [1967]. Beneš, Shepp and Witsenhausen [1979] proved optimality of the bang-bang law in the case of a quadratic running cost on the state and no cost on the control. They also treated the finite-fuel problem with a discounted cost criterion.

cont In the present paper we proceed by formulating the control problem. ~~in Section 2.~~ The Bellman equation of dynamic programming is explicitly solved ~~in Section 3~~ and the candidate for the optimal law ~~u^* in (1.1)~~ is discerned from the properties of the solution. Optimality of the candidate is proved. ~~in Section 4~~.

KEY WORDS AND PHRASES: Discounted stochastic control, Bellman equation, dead-zone controllers

2. THE STOCHASTIC CONTROL PROBLEM

Consider as basic probability space Ω the space $C(\mathbb{R}^+)$ of continuous, real-valued functions on $\mathbb{R}^+$ and let $\mathcal{F}_t$, $t \geq 0$ denote the σ-field generated by $\{x_s;\ s \leq t\}$, $x \in \Omega$. Consider also the σ-field $\mathcal{M}$ generated by the subsets M of $\mathbb{R}^+ \times C(\mathbb{R}^+)$ with the property that each t-section M_t of M belongs to $\mathcal{F}_t$ and each x-section M_x of M is Lebesgue measurable. A function g defined on $\mathbb{R}^+ \times C(\mathbb{R}^+)$ is $\mathcal{M}$-measurable if and only if $g(t,\cdot)$ is $\mathcal{F}_t$-measurable for any $t \geq 0$ and $g(\cdot,x)$ is Lebesgue measurable, for any $x \in C(\mathbb{R}^+)$.

Definition 2.1: Let the control measure space be the compact interval $[-1,1]$ with its Borel sets. An admissible nonanticipative control u is a measurable function $u: (\mathbb{R}^+ \times C(\mathbb{R}^+)) \to [-1,1]$.

The class of all such controls is denoted by $\mathcal{U}$. For any control law $u \in \mathcal{U}$ and any $x \in \mathbb{R}$ we can construct by means of the Girsanov theorem a probability space $(\Omega, \mathcal{F}, P)$ and a pair of stochastic processes (x_t, w_t) on it, such that $\{w_t;\ t \geq 0\}$ is a Wiener process with respect to P and the stochastic differential equation

$$(2.1) \qquad dx_t = u_t(x)dt + dw_t, \qquad t \geq 0$$

$$(2.2) \qquad x_0 = x$$

is satisfied. Such a "weak solution" of (2.1) is known to be unique in the sense of the probability law; see, for instance, Liptser and Shiryayev [1977].

Consider now a nonnegative function ϕ on the reals which is even, $C^2(\mathbb{R})$, uniformly convex in the sense that

$$(2.3) \qquad 0 < k \leq \ddot{\phi}(x) \leq K, \quad \text{all} \quad x \in \mathbb{R}$$

for some positive constants k, K, with $\ddot{\phi}(x)$ decreasing on $x > 0$. The <u>control problem</u> consists in finding a law $u^* \in \mathcal{U}$ that minimizes the "discounted" expected total cost

$$(2.4) \qquad J(x;u) = E \int_0^{\infty} e^{-\alpha t}(|u_t(x)| + \phi(x_t))dt$$

of starting at place x and using control u, over all $u \in \mathcal{U}$, $x \in \mathbb{R}$. Here E denotes expectation with respect to the probability measure P, $\alpha > 0$ is the "discount factor", $\phi(\cdot)$ is the running cost on the state and $|\cdot|$ is the cost of control.

3. THE EQUATION OF DYNAMIC PROGRAMMING

The method proceeds by constructing a solution to the Bellman equation of dynamic programming which satisfies certain growth and symmetry conditions. Introduce the function

$$\text{(3.1)} \qquad a(p) = \min_{|u|\le 1} (|u|+pu) = \begin{cases} 0, & |p| \le 1 \\ 1-|p|, & |p| \ge 1. \end{cases}$$

The formal Bellman equation for this problem is

$$\text{(3.2)} \qquad \alpha v = \frac{1}{2} v_{xx} + a(v_x) + \phi(x), \qquad x \in \mathbb{R}.$$

We are looking for a positive number b and an even solution of (3.2) $v(x) = O(x^2)$ as $|x| \to \infty$, *such that* $v_x(b) = 1$ and

$$\text{(3.3)} \qquad \alpha v = \frac{1}{2} v_{xx} + \phi(x), \qquad 0 < v\ (x) < 1 \quad \text{on} \quad 0 < x < b$$

$$\text{(3.4)} \qquad \alpha v = \frac{1}{2} v_{xx} + 1 - v_x + \phi(x), \quad v_x(x) > 1 \quad \text{on} \quad x > b.$$

A particular solution to the equation in (3.3) is given by the cost of "doing nothing" all the time. Indeed, consider the "naive" control law $u_t(x) \equiv 0$. The corresponding cost is

$$p(x) = E\int_0^\infty e^{-\alpha t}\phi(x+w_t)dt$$

and it becomes an easy exercise in Laplace transforms to verify that

$$(3.5) \qquad p(x) = \frac{1}{\sqrt{2\alpha}}\left[e^{-x\sqrt{2\alpha}}\int_{-\infty}^{x}\phi(z)e^{z\sqrt{2\alpha}}dz + \int_{x}^{\infty}\phi(z)e^{-z\sqrt{2\alpha}}dz\right].$$

This function is even, has the growth of $\phi(\cdot)$ as $|x| \to \infty$ and satisfies the equation in (3.3) as is easily verified. To get the general solution of the latter, we add to $p(x)$ a solution

$$A_1 e^{x\sqrt{2\alpha}} + A_2 e^{-x\sqrt{2\alpha}}$$

of the homogeneous $\alpha v = \frac{1}{2} v_{xx}$. Since $v(\cdot)$ has to be even, and consequently $v_x(0) = 0$, $A_1 = A_2 = \frac{A}{2}$. So

$$(3.6) \qquad v(x) = A\cosh(x\sqrt{2\alpha}) + p(x), \quad \text{on} \quad 0 < x < b.$$

Condition $v_x(b) = 1$ then implies

$$(3.7) \qquad A = \frac{p'(b) - 1}{\sqrt{2\alpha}\cdot\sinh(b\sqrt{2\alpha})} .$$

Similarly, a particular solution to (3.4) is obtained by considering the cost corresponding to the naive law $u_t(x) \equiv -1$ of pushing with full force to the left all the time:

$$q(x) \triangleq E\int_0^\infty e^{-\alpha t}(1 + \phi(x-t+w_t))dt$$

and is easily verified that, if $\beta \triangleq \sqrt{1+2\alpha} - 1$,

$$(3.8)\quad q(x) = \frac{1}{\alpha} + \frac{1}{1+\beta}\left[e^{-\beta x}\int_{-\infty}^{x}\phi(z)e^{\beta z}dz + e^{(2+\beta)x}\int_{x}^{\infty}\phi(z)e^{-(2+\beta)z}dz\right],$$

solves the equation in (3.4) and has the growth of $\phi(\cdot)$ as $|x| \to \infty$. To get the general solution of (3.4) one has to add to $q(x)$ the general solution

$$Be^{-\beta x} + B_1 e^{(2+\beta)x}$$

of the corresponding homogeneous equation $\alpha v = \frac{1}{2} v_{xx} - v_x$ (note that $2+\beta$ and $-\beta$ are the roots of the characteristic polynomial $s^2 - 2s - 2\alpha$). The growth condition implies $B_1 = 0$, so

$$(3.9)\qquad v(x) = Be^{-\beta x} + q(x), \quad \text{on} \quad x > b,$$

where

$$(3.10)\qquad B = \frac{q'(b)-1}{\beta}e^{\beta b}$$

because of $v_x(b) = 1$. Matching the values of $v(\cdot)$ from the two sides at $x = b$ gives the equation for the switching point b:

$$(3.11)\qquad \tanh(b\sqrt{2\alpha}) = -\frac{1}{\sqrt{2\alpha}}\,\frac{p'(b)-1}{\frac{q'(b)-1}{\beta} + q(b) - p(b)},$$

$p(\cdot)$ and $q(\cdot)$ being the functions in (3.5), (3.8).

<u>Proposition 3.1</u>. There is exactly one positive solution b to equation (3.11).

<u>Proof</u>. A lot of simple calculus shows that $p'(x) - 1 = \frac{1}{\alpha} m_1(x)$ and $\frac{q'(x) - 1}{\beta} + q(x) - p(x) = \frac{\beta}{2\alpha^2} m_2(x)$, where

$$m_1(x) \triangleq \phi'(x) - \alpha + \frac{1}{2}\left\{e^{x\sqrt{2\alpha}} \int_x^\infty \phi''(z) e^{-z\sqrt{2\alpha}} dz \right.$$

$$\left. - e^{-x\sqrt{2\alpha}} \int_{-\infty}^x \phi''(z) e^{z\sqrt{2\alpha}} dz \right\}$$

$$m_2(x) \triangleq \phi'(x) - \alpha + e^{(2+\beta)x} \int_x^\infty \phi''(z) e^{-(2+\beta)z} dz -$$

$$- \frac{\sqrt{2\alpha}}{\beta}\left\{e^{x\sqrt{2\alpha}} \int_x^\infty \phi''(z) e^{-z\sqrt{2\alpha}} dz + e^{-x\sqrt{2\alpha}} \int_\infty^x \phi''(z) e^{z\sqrt{2\alpha}} dz \right\}$$

so that equation (3.11) becomes: $\tanh(x\sqrt{2\alpha}) = \frac{\sqrt{2\alpha}}{\beta} m(x)$, $m(x) \triangleq -\frac{m_1(x)}{m_2(x)}$.

Note that $m_1(0) = -\alpha < 0$ and that $m_1(x)$ is strictly increasing to infinity as $x \uparrow \infty$, since

$$m_1'(x) = \sqrt{\frac{\alpha}{2}}\left[e^{x\sqrt{2\alpha}} \int_x^\infty \phi''(z) e^{-z\sqrt{2\alpha}} dz + e^{-x\sqrt{2\alpha}} \int_{-\infty}^x \phi''(z) e^{z\sqrt{2\alpha}} dz\right]$$
$$\geq k > 0.$$

Thus, there exists a unique number $b_1 > 0$, such that $m_1(b_1) = 0$. On the other hand, since $\beta < \sqrt{2\alpha} < 2 + \beta$, $m_1(x) > m_2(x)$ and

$$m_2'(x) = \frac{\alpha}{\beta}\left[2\int_x^{\infty}\phi''(z)e^{-(2+\beta)(z-x)}dz + \int_0^{\infty}(\phi''(z-x) - \phi''(z+x))e^{-z\sqrt{2\alpha}}\,dz\right] \geqq k > 0$$

on $x > 0$, by the assumption of decreasing curvature. So there exists a unique number $b_2 > b_1$, such that $m_2(b_2) = 0$. Now the function $m(x)$ is negative on $(0,b_1)$ and (b_2,∞), is equal to zero at b_1, and increases montonically to infinity on (b_1,b_2) as $x \uparrow b_2$.

Consequently, there exists a unique $b \in (b_1,b_2)$ such that: $\tanh(b\sqrt{2\alpha}) = \frac{\sqrt{2\alpha}}{\beta}\, m(b)$, q.e.d.

Once b has been thus determined, one constructs the function

$$(3.12) \quad v(x) = \frac{1 - p'(b)}{\sqrt{2\alpha}\cdot\sinh(b\sqrt{2\alpha})}\cosh(x\sqrt{2\alpha}) + p(x) \;;\quad 0 \leq x \leq b$$

$$= \frac{q'(b) - 1}{\beta}\, e^{-\beta(x-b)} + q(x) \;;\quad x > b$$

$$= v(-x) \;;\quad x < 0$$

in accordance with (3.6), (3.7), (3.9), (3.10), where $p(x)$ and $q(x)$ are again the functions in (3.5) and (3.8). The function $v(\cdot)$ in (3.12) satisfies equations (3.3) and (3.4) on (a,b) and (b,∞) respectively, as well as $v_x(b) = 1$, by construction, and $v(b_+) = v(b_-)$, by (3.11).

It remains to prove that $v(\cdot)$ solves the Bellman equation (3.2). Suffices to prove: $0 \leq v_x(x) \leq 1$, on $[0,b]$ and

$v_x(x) \geq 1$, on $[b,\infty)$, while in turn this is an easy corollary of convexity:

<u>Proposition 3.2</u>. The function in (3.12) is convex: $v_{xx}(x) \geq 0$, $x \in \mathbb{R}$.

<u>Proof</u>. A bit of algebra shows that, on $0 \leq x \leq b$,

$$\frac{1}{2} v_{xx}(x) = \alpha v(x) - \phi(x) \geq \frac{\beta}{2\alpha}\left[\phi'(b) + e^{(2+\beta)b}\int_b^{\infty}\phi''(z)e^{-(2+\beta)z}dz\right]$$

$$+ (\alpha p(x) - \phi(x)) - (\alpha p(b) - \phi(b)).$$

Note that: $\alpha p'(x) - \phi'(x) = \frac{1}{2}\int_0^{\infty}(\phi''(z+x) - \phi''(z-x))e^{-z\sqrt{2\alpha}} \leq 0$, by the decreasing second derivative assumption. Therefore, $\alpha p(x) - \phi(x) \geq \alpha p(b) - \phi(b)$ and

$$v_{xx}(x) \geq \frac{\beta k(b+1)}{\alpha}, \quad 0 \leq x \leq b. \tag{3.13}$$

By continuity of $v_{xx}(\cdot)$; $v_{xx}(x) > 0$ on $(b,b+\varepsilon)$, $\varepsilon > 0$ sufficiently small. On the other hand, if $w = v_{xx}$:

$$w_{xx} - 2w_x - 2\alpha w = -2\phi''(x) \leq 0, \quad \text{on} \quad (b,\infty).$$

By the maximum principle (Friedman [1964], p. 53, Theorem 18), $v_{xx}(\cdot)$ cannot have a negative minimum on (b,∞). However, on this interval,

$$v_{xx}(x) = \beta(q'(b)-1)e^{-\beta(x-b)} + q''(x) \geq \beta(q'(b)-1)e^{-\beta(x-b)} + \frac{k}{\alpha},$$

since

$$q''(x) = \frac{1}{1+\beta}\left[e^{(2+\beta)x}\int_x^\alpha \phi''(z)e^{-(2+\beta)z}dz + e^{-\beta x}\int_{-\infty}^x \phi''(z)e^{\beta z}dz\right].$$

Therefore, $v_{xx}(x) > 0$ for x sufficiently large, so if $v_{xx}(\bar{x}) < 0$, some $\bar{x} \in (b,\infty)$, $v_{xx}(\cdot)$ would have a negative minimum there, contradicting the maximum principle. Therefore $v_{xx}(x) \geqq 0$, also on $[b+\varepsilon,\infty)$, q.e.d.

<u>Special Case</u>: In the special case $\phi(x) = x^2$, we have

$$p(x) = \frac{x^2}{\alpha} + \frac{1}{\alpha^2}, \quad q(x) = \frac{x^2}{\alpha} - \frac{2}{\alpha}x + \frac{\alpha^2+\alpha+2}{\alpha^2},$$

$m_1(x) = x - \frac{\alpha}{2}$, $m_2(x) = x - (\frac{\alpha}{2} + \frac{1}{\alpha})$, so $\frac{\alpha}{2} < b < \frac{\alpha}{2} + \frac{1}{\alpha}$. It can be shown that $v_{xx}(x) \geqq \frac{2(\alpha-\beta)}{\alpha^2} > 0$, any $x \in \mathbb{R}$, in this case.

4. THE OPTIMAL LAW

Let us prove that $v(\cdot)$ is the value function of the control problem, i.e. an (attainable) lower bound on the expected discounted total cost, and try to discern the law that achieves this infimum. Consider any admissible law $u \in \mathcal{U}$ along with the corresponding state process (x_t^u), solving equation (2.1)-(2.2) in the weak sense. We introduce the process

$$V_t^u \triangleq v(x_t^u)e^{-\alpha t} \tag{4.1}$$

and note that

$$Ev(x_t^u) \leqq Ev(|x|+t+|w_t|) = O(t^2) \quad \text{as} \quad t \to \infty, \text{ so} \quad \lim_{t\to\infty} EV_t^u = 0,$$

any $\alpha > 0$, $x \in \mathbb{R}$, $u \in \mathcal{U}$.

Applying Itô's rule to (4.1) we get

$$V_T^u = v(x) + \int_0^T e^{-\alpha t}(-\alpha v(x_t^u) + \tfrac{1}{2} v_{xx}(x_t^u) + u_t v_x(x_t^u) + |u_t| + \phi(x_t^u))dt - \int_0^T e^{-\alpha t}(|u_t(x^u)| + \phi(x_t^u))dt + \int_0^T e^{-\alpha t} v_x(x_t^u)dw_t.$$

Since $v(\cdot)$ satisfies equation (3.2), the first integrand is nonnegative. Taking expectations and then passing to the limit as $T \to \infty$ we get

$$J(x;u) = E\int_0^\infty e^{-\alpha t}(|u_t(x^u)| + \phi(x_t^u))dt \geqq v(x); \; u \in \mathcal{U}, \; x \in \mathbb{R}. \tag{4.2}$$

Consider now the law $u^* \in \mathcal{U}$

(1.1) $$u_t^*(x) = \begin{cases} -1, & x_t > b \\ 0, & |x_t| \leq b \\ 1, & x_t < -b \end{cases}$$

obtained through the minimization: $|u_t^*(x)| + v_x(x_t)\cdot u_t^*(x) = a(v_x(x_t))$. The corresponding state process (x_t^*) satisfies

(4.4) $$dx_t^* = u_t^*(x^*)dt + dw_t; \quad t \geq 0$$
$$x_0^* = x$$

on an appropriate probability space. Although no explicit use is made of this fact in the present context, we mention that (4.4) is strongly solvable for $x^*_.$ as a causal functional of $w_.$ because $u_t^*(x)$ in (4.3) is instantaneous, bounded and measurable; see Zvonkin [1974]. Then the inequalities above hold as equalities, and

(4.5) $$J(x;u^*) = v(x); \quad x \in \mathbb{R}.$$

From (4.2), (4.5), $v(x)$ is a lower bound on the performance (2.4), and is achieved by the process (x_t^*). In other words, $u_t^*(x)$ is optimal.

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